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STABILITY AND CONVERGENCE OF A GENERALIZED CRANK-NICOLSON SCHEM--ETC(U)
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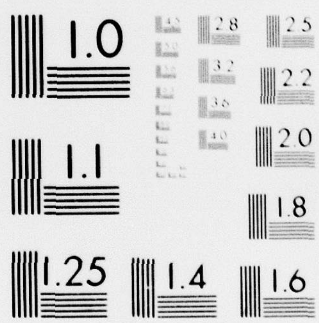

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STABILITY AND CONVERGENCE OF A GENERALIZED CRANK-NICOLSON SCHEME ON A VARIABLE MESH FOR THE HEAT EQUATION

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ABSTRACT

In a previous paper devoted to the numerical solution of the Stefan problem, the author has proposed a numerical scheme to solve the heat equation on a variable mesh; this scheme is a generalization of the classical Crank-Nicolson scheme since it is identical to the Crank-Nicolson scheme in the particular case of a fixed mesh. Numerical experiments have been performed in one and two space-dimensions, but no mathematical results had been proved. In the present paper, the stability and convergence of the scheme are established together with an error estimate.

AMS (MOS) Subject Classification - 65M05, 65M10, 65M15.

Key Words - Heat equation, Moving boundary, Variable mesh, Finite elements, Stability, Convergence, Error estimate

Work Unit Number 7 - Numerical Analysis

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SIGNIFICANCE AND EXPLANATION

Moving boundary problems occur in many areas of physics and engineering. The best known example is the Stefan problem which represents the melting of a solid: the moving boundary is the interface between the solid and the liquid; in each phase, the temperature satisfies the heat equation. An efficient way of solving moving boundary problems is to use a variable mesh. In a previous paper, such a method has been proposed for the heat equation and numerical experiments have been performed. In this paper, the stability and convergence of this method are proved.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report-

STABILITY AND CONVERGENCE OF A GENERALIZED CRANK-NICOLSON SCHEME ON A VARIABLE MESH FOR THE HEAT EQUATION

Pierre Jamet*

1) INTRODUCTION

Let $x \in \mathbb{R}$ be the space variable and t the time. Let $s_1(t)$ and $s_2(t)$ be two given continuous functions defined for $t \geq 0$, such that $s_1(t) < s_2(t)$ for all t. Let $f(x,t)$ be a given function defined for $s_1(t) < x < s_2(t)$, $t > 0$ and let $u^0(x)$ be a given function defined for $s_1(0) < x < s_2(0)$.

We want to solve the following problem.

Problem (P): Find a function $u = u(x,t)$ such that

1.1) $$Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f, \quad \text{for } s_1(t) < x < s_2(t),\ t > 0 ,$$

1.2) $$u(x,0) = u^0(x), \quad \text{for } s_1(0) < x < s_2(0) ,$$

1.3) $$u(s_1(t)) = u(s_2(t)) = 0, \quad \text{for } t > 0 .$$

We will assume that the functions $s_1(t)$ and $s_2(t)$ satisfy the two following hypotheses.

1.4) $$s_2(t) - s_1(t) \geq \ell_0 > 0, \quad \text{for all } t > 0 ,$$

1.5) $|s_j(t') - s_j(t)| \leq c_0|t' - t|$, for all t and t' (Lipshitz continuity), with $j = 1$ or 2, where ℓ_0 and c_0 are constants.

The numerical scheme

We consider a mesh arranged in the following way. Let i be a space index, $0 \leq i \leq I$, and $n \geq 0$ be a time index. The mesh-points are denoted by $P_i^n = (x_i^n, t^n)$, where $t_0 = 0$, $t^n < t^{n+1}$, $x_0^n = s_1(t^n)$, $x_i^n < x_{i+1}^n$, $x_I^n = s_2(t^n)$ (see Figure 1).

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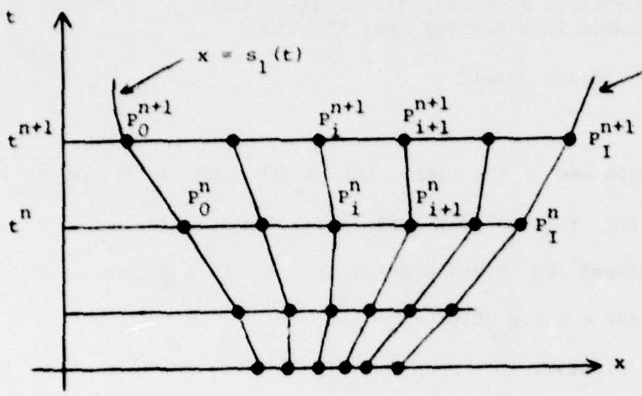

Figure 1

We want to compute a function u_h which approximates the solution u of problem (P) at the mesh-points P_i^n. We will denote $u_i^n = u_h(P_i^n)$. According to the method of Bonnerot and Jamet [1], these values are determined by solving the following problem.

Discrete problem (P_h)

Find $\{u_i^n\}$, $0 \le i \le I$, $n \ge 0$, such that

$$
(1.6)\quad \left\{
\begin{aligned}
&(x_{i+1}^{n+1} - x_{i-1}^{n+1})u_i^{n+1} - (x_{i+1}^{n} - x_{i-1}^{n})u_i^{n} + \\
&- \frac{1}{2}(x_{i+1}^{n+1} - x_{i+1}^{n})(u_{i+1}^{n} + u_{i+1}^{n+1}) + \frac{1}{2}(x_{i-1}^{n+1} - x_{i-1}^{n})(u_{i-1}^{n} + u_{i-1}^{n+1}) + \\
&- (t^{n+1} - t^{n})\left(\frac{u_{i+1}^{n} - u_i^{n}}{x_{i+1}^{n} - x_i^{n}} + \frac{u_{i+1}^{n+1} - u_i^{n+1}}{x_{i+1}^{n+1} - x_i^{n+1}} - \frac{u_i^{n} - u_{i-1}^{n}}{x_i^{n} - x_{i-1}^{n}} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{x_i^{n+1} - x_{i-1}^{n+1}}\right) = \\
&= \frac{1}{2}(t^{n+1} - t^{n})((x_{i+1}^{n} - x_{i-1}^{n})f(P_i^{n}) + (x_{i+1}^{n+1} - x_{i-1}^{n+1})f(P_i^{n+1})),
\end{aligned}
\right.
$$

for $1 \le i \le I-1$ and $n \ge 0$,

$$(1.7)\quad u_i^0 = u^0(x_i^0), \quad \text{for } 1 \le i \le I-1,$$

$$(1.8)\quad u_0^n = u_I^n = 0, \quad \text{for } n \ge 0.$$

After division by $\frac{1}{2}(t^{n+1} - t^n)((x^n_{i+1} - x^n_{i-1}) + (x^{n+1}_{i+1} - x^{n+1}_{i-1}))$, we can write (1.6) in the form

$$(L_h u_h)^n_i = \tilde{f}^n_i , \tag{1.9}$$

where L_h is a discrete operator which approximates the differential operator L, as will be shown later (see lemma 3.1), and $\tilde{f}^n_i$ is a weighted average of $f(P^n_i)$ and $f(P^{n+1}_i)$.

In this paper, we will assume that the nodes P^n_i are equally spaced for each n, i.e.

$$x^n_{i+1} - x^n_i = h^n = (s_2(t^n) - s_1(t^n))/I, \quad \text{for} \quad 0 \leq i \leq I - 1 . \tag{1.10}$$

As already noted in [1], in the particular case of a fixed mesh we have $h^{n+1} = h^n = h$ and $x^{n+1}_i - x^n_i = 0$, for all i; hence, the relation (1.6) yields the classical Crank-Nicolson scheme.

Although successful numerical experiments, as well as practical computations for moving boundary problems of Stefan type, have been performed by means of the foregoing method, no mathematical results had been proved yet. In section 2 of this paper, we prove the stability of the method; in section 3, we establish an error estimate and prove the convergence; finally, section 4 is devoted to several remarks and comments relating the present method with other methods which have been studied to solve the same problem.

2) STABILITY

We will use the following notation.

$$\|\varphi^n\|_h = \Big(\frac{1}{I-1}\sum_{i=1}^{I-1} |\varphi(P_i^n)|^2\Big)^{1/2},$$

for any function φ defined at the nodes P_i^n for $1 \leq i \leq I-1$. Also, we will note $k^n = t^{n+1} - t^n$.

Theorem 2.1

Assume

2.1) $$k^n \leq \lambda \min\{h^n, h^{n+1}\}, \quad \text{for all} \quad n \geq 0,$$

where λ is a positive constant. Then, there exists three constants κ, γ and γ' which depend only on c_0, ℓ_0 and λ such that:

i) The discrete problem (P_h) admits a unique solution $u_h = \{u_i^n\}$ provided $k^n \leq \kappa$ for all $n \geq 0$.

ii) The following estimate holds

2.2) $$\|u_h^n\|_h^2 \leq e^{\gamma t^n}\|u^0\|_h^2 + \gamma' e^{\gamma t^n}\Big(\sum_{\nu=0}^{n-1} (t^{\nu+1} - t^\nu)\|f^\nu\|_h^2\Big),$$

for all $n > 0$.

Remark: Because of assumption (2.1), the condition $k^n \leq \kappa$ is automatically satisfied for h^n small enough.

Proof: For simplicity, we will omit the subscript h in the expression of the discrete norm $\|\cdot\|_h$.

First, we write two inequalities which will be useful later. Hypotheses (1.4) and (1.5) imply

2.3) $$|h^{n+1} - h^n| \leq ck^n \min\{h^n, h^{n+1}\}, \quad \text{with} \quad c = 2c_0/\ell_0.$$

Hypotheses (1.4), (1.10) and (2.1) imply

2.4) $$|x_i^{n+1} - x_i^n| \leq c' \min\{h^n, h^{n+1}\}, \quad \text{for all} \quad i,$$

with $c' = c_0\lambda$.

Note that a condition similar to (2.3) has been used by Lesaint and Raviart [9].

Now, we multiply (1.6) by $(h^{n+1}u_i^n + h^n u_i^{n+1})$ and sum for $1 \le i \le I-1$. We get

(2.5) $$A_1 + A_2 + A_3 = A_4 \text{ , with}$$

$$A_1 = 2\sum_{i=1}^{I-1} (h^{n+1}u_i^{n+1} - h^n u_i^n)(h^{n+1}u_i^n + h^n u_i^{n+1}) ,$$

$$A_2 = -\frac{1}{2}\sum_{i=1}^{I-1} (x_{i+1}^{n+1} - x_{i+1}^n)(u_{i+1}^n + u_{i+1}^{n+1})(h^{n+1}u_i^n + h^n u_i^{n+1}) +$$

$$+\frac{1}{2}\sum_{i=1}^{I-1} (x_{i-1}^{n+1} - x_{i-1}^n)(u_{i-1}^n + u_{i-1}^{n+1})(h^{n+1}u_i^n + h^n u_i^{n+1}) ,$$

$$A_3 = -k^n \sum_{i=1}^{I-1} \left(\frac{u_{i+1}^n - u_i^n}{h^n} - \frac{u_i^n - u_{i-1}^n}{h^n}\right)(h^{n+1}u_i^n + h^n u_i^{n+1}) +$$

$$-k^n \sum_{i=1}^{I-1} \left(\frac{u_{i+1}^{n+1} - u_i^{n+1}}{h^{n+1}} - \frac{u_i^{n+1} - u_{i-1}^{n+1}}{h^{n+1}}\right)(h^{n+1}u_i^n + h^n u_i^{n+1}) ,$$

$$A_4 = k^n(h^n + h^{n+1}) \sum_{i=1}^{I-1} \tilde{f}_i^n (h^{n+1}u_i^n + h^n u_i^{n+1}) .$$

We will consider separately each of the terms A_1, A_2, A_3 and A_4.

Estimation of A_1: We have

$$A_1 = 2h^n h^{n+1}\left(\sum_{i=1}^{I-1} (u_i^{n+1})^2 - \sum_{i=1}^{I-1} (u_i^n)^2\right) + 2((h^{n+1})^2 - (h^n)^2) \sum_{i=1}^{I-1} u_i^n u_i^{n+1} .$$

Because of (2.3), we have

(2.6) $$|(h^{n+1})^2 - (h^n)^2| = |h^{n+1} - h^n|(h^n + h^{n+1}) \le$$
$$\le ck^n(h^n + h^{n+1})\min\{h^n, h^{n+1}\} \le 2ck^n h^n h^{n+1} .$$

Hence, by Cauchy's inequality:

(2.7) $$A_1 \ge 2h^n h^{n+1}(1 - ck^n)(I-1)\|u_h^{n+1}\|^2 +$$
$$- 2h^n h^{n+1}(1 + ck^n)(I-1)\|u_h^n\|^2 .$$

Estimation of A_2: Shifting the index i in the second summation of the expression of A_2 and taking account of the boundary conditions (1.8), we get:

$$A_2 = -\frac{1}{2}\sum_{i=1}^{I-1} (x_{i+1}^{n+1} - x_{i+1}^{n})(u_{i+1}^{n} + u_{i+1}^{n+1})(h^{n+1}u_i^n + h^n u_i^{n+1}) +$$

$$+\frac{1}{2}\sum_{i=1}^{I-1} (x_i^{n+1} - x_i^n)(u_i^n + u_i^{n+1})(h^{n+1}u_{i+1}^n + h^n u_{i+1}^{n+1}) .$$

But

$$h^{n+1}u_i^n + h^n u_i^{n+1} = \frac{1}{2}(h^n + h^{n+1})(u_i^n + u_i^{n+1}) - \frac{1}{2}(h^{n+1} - h^n)(u_i^{n+1} - u_i^n)$$

and

$$(x_{i+1}^{n+1} - x_{i+1}^n) - (x_i^{n+1} - x_i^n) = h^{n+1} - h^n .$$

Hence, we can write

$$A_2 = -\frac{1}{4}(h^{n+1} - h^n)(h^n + h^{n+1})\sum_{i=1}^{I-1} (u_i^n + u_i^{n+1})(u_{i+1}^n + u_{i+1}^{n+1}) +$$

$$+\frac{1}{4}(h^{n+1} - h^n)\sum_{i=1}^{I-1} (x_{i+1}^{n+1} - x_{i+1}^n)(u_{i+1}^n + u_{i+1}^{n+1})(u_i^{n+1} - u_i^n) +$$

$$-\frac{1}{4}(h^{n+1} - h^n)\sum_{i=1}^{I-1} (x_i^{n+1} - x_i^n)(u_i^n + u_i^{n+1})(u_{i+1}^{n+1} - u_{i+1}^n) .$$

Applying (2.6) and the inequality

$$|h^{n+1} - h^n||x_i^{n+1} - x_i^n| \le cc'k^n h^n h^{n+1} ,$$

which follows from (2.3) and (2.4), we get

$$(2.8) \quad |A_2| \le \frac{1}{4} ck^n h^n h^{n+1}\Bigg[2\sum_{i=1}^{I-1} |u_i^n + u_i^{n+1}||u_{i+1}^n + u_{i+1}^{n+1}| +$$

$$+ c'\sum_{i=1}^{I-1} |u_{i+1}^n + u_{i+1}^{n+1}||u_i^{n+1} - u_i^n| + c'\sum_{i=1}^{I-1} |u_i^n + u_i^{n+1}||u_{i+1}^{n+1} - u_{i+1}^n|\Bigg] .$$

Apply the inequality

$$(a_1 + a_2)(a_3 + a_4) \le \sum_{j=1}^{4} (a_j)^2, \quad \text{for all real numbers } a_j.$$

We get

$$\sum_{i=1}^{I-1} |u_i^n + u_i^{n+1}||u_{i+1}^n + u_{i+1}^{n+1}| \le 2(I-1)(\|u_h^n\|^2 + \|u_h^{n+1}\|^2) ,$$

and the same estimate is valid for the two other sums in the right hand side member of (2.8). Hence,

$$|A_2| \le c(1 + c')k^n h^n h^{n+1}(I - 1)(\|u_h^n\|^2 + \|u_h^{n+1}\|^2) . \tag{2.9}$$

Estimation of A_3:

Let $v_i^n = \dfrac{u_i^n}{h^n} + \dfrac{u_i^{n+1}}{h^{n+1}}$. Then,

$$A_3 = -k^n h^n h^{n+1} \sum_{i=1}^{I-1} ((v_{i+1}^n - v_i^n) - (v_i^n - v_{i-1}^n))v_i^n =$$

$$= k^n h^n h^{n+1} \sum_{i=0}^{I-1} (v_{i+1}^n - v_i^n)^2 =$$

$$= k^n h^n h^{n+1} \sum_{i=0}^{I-1} \left(\frac{u_{i+1}^n - u_i^n}{h^n} + \frac{u_{i+1}^{n+1} - u_i^{n+1}}{h^{n+1}} \right)^2 .$$

Hence,

$$A_3 \ge k^n h^n h^{n+1}(I - 1)\|\delta_x^{(n)} u_h^n + \delta_x^{(n+1)} u_h^{n+1}\|^2 , \tag{2.10}$$

where $\delta_x^{(n)}$ denotes the forward difference quotient at the time step n, i.e. $\delta_x^{(n)} u_h^n$ is the vector with components $(u_{i+1}^n - u_i^n)/h^n$.

Estimation of A_4: We have

$$|A_4| \le k^n(h^n + h^{n+1})(I - 1)(h^{n+1}\|u_h^n\| + h^n\|u_h^{n+1}\|)\|\tilde{f}^n\| .$$

But, (2.3) implies

$$h^n + h^{n+1} \le (2 + ck^n)\min(h^n, h^{n+1}) .$$

Hence

$$|A_4| \le k^n(2 + ck^n)h^n h^{n+1}(I - 1)(\|u_h^n\| + \|u_h^{n+1}\|)\|\tilde{f}^n\| .$$

Finally

$$|A_4| \le \frac{1}{2} k^n(2 + ck^n)h^n h^{n+1}(I - 1)(\|u_h^n\|^2 + \|u_h^{n+1}\|^2 + 2\|\tilde{f}^n\|^2) . \tag{2.11}$$

Taking (2.7), (2.9), (2.10) and (2.11) into (2.5) and dividing by $2h^n h^{n+1}(I - 1)$, we get:

$$(2.12)\qquad (1 - \alpha^n k^n)\|u_h^{n+1}\|^2 + \frac{1}{2} k^n \|\delta_x^{(n)} u_h^n + \delta_x^{(n+1)} u_h^{n+1}\|^2 \leq$$

$$\leq (1 + \alpha^n k^n)\|u_h^n\|^2 + \beta^n k^n \|\bar{f}^n\|^2, \text{ with}$$

$$\alpha^n = c + \frac{1}{2} c(1 + c') + \frac{1}{4}(2 + ck^n),$$

$$\beta^n = \frac{1}{2}(2 + ck^n).$$

Assume that the time increments k^n are uniformly bounded; for example, assume $k^n \leq 2$. Then, we can replace α^n and β^n in (2.12) by two constants $\alpha = \frac{1}{2}(4c + cc' + 1)$ and $\beta = 1 + c$.

Let $\gamma > 2\alpha$, for example $\gamma = 3\alpha$. Then, there exists a constant $\tilde{\kappa} > 0$ which depends on α and γ such that

$$1 - \alpha y \geq e^{-\gamma y/2}, \quad \text{for } 0 \leq y \leq \tilde{\kappa}.$$

Let $\kappa = \min\{2, \tilde{\kappa}\}$. Then, for $k^n \leq \kappa$, we have

$$\|u_h^{n+1}\|^2 \leq e^{\gamma(t^{n+1} - t^n)}\|u_h^n\|^2 + \gamma'(t^{n+1} - t^n)\|\bar{f}^n\|^2,$$

with $\gamma' = \beta e^{\gamma\kappa/2}$

The estimate (2.2) follows by mathematical induction. The constants $\alpha, \beta, \gamma, \kappa$ and γ' depend on c and c', i.e. on c_0, ℓ_0 and λ. The estimate (2.2) implies the uniqueness and therefore the existence of the solution u_h of the system of linear algebraic equations (1.6), (1.7), (1.8). Therefore, the Theorem 2.1 is proved.

3) ERROR ESTIMATE AND CONVERGENCE

Let T be the final time at which we want to stop the computations and let R be the corresponding domain in the (x,t)-plane, i.e. $R = \{(x,t);\ s_1(t) < x < s_2(t),\ 0 < t < T\}$. For any function $\varphi \in C^4(\bar{R})$, let

$$|\varphi|_{m,\infty,R} = \operatorname*{Max}_{|j|=m} \ \operatorname*{Sup}_{(x,t)\in R} |D^j\varphi(x,t)| ,$$

where D^j denotes an arbitrary derivative of order m with respect to x and t, $0 \leq m \leq 4$.

Let $h = \operatorname*{Max}_{t^n \leq T} \{h^n\}$ and $k = \operatorname*{Max}_{t^n \leq T} \{k^n\}$, where the maximum is taken for all $n \geq 0$ such that $t^n \leq T$. The squares of h and k will be denoted by h^2 and k^2 respectively; they should not be confused with the values of h^n and k^n for $n = 2$ which are needed nowhere in the paper.

Let L_h be the discrete operator defined in (1.9) i.e. such that

$$(L_h u_h)_i^n = (M_h u_h)_i^n / k^n (h^n + h^{n+1}),$$

where $(M_h u_h)_i^n$ is the left hand side member of (1.6).

Lemma 3.1 (Truncation error)

Let the assumptions be the same as in Theorem 2.1. Let P_i^n and P_i^{n+1} be two mesh-points, $1 \leq i \leq I - 1$, such that the straight segment which joins them is contained in R. Then

$$|(L_h\varphi)_i^n - (L\varphi)_i^{n+1/2}| \leq C_1(h^2 + k^2) \operatorname*{Max}_{2\leq m\leq 4} |\varphi|_{m,\infty,R},$$

for all functions $\varphi \in C^4(\bar{R})$, where $(L\varphi)_i^{n+1/2}$ is the value of $L\varphi$ at the mid-point $P_i^{n+1/2} = \frac{1}{2}(P_i^n + P_i^{n+1})$ and C_1 is a constant which depends on c_0, ℓ_0 and λ.

Proof: Let $x_i^{n+1} - x_i^n = d_i^n$. Then, by a Taylor expansion at the point $P_i^{n+1/2}$, we get

3.2) $$(M_h\varphi)_i^n = k^n(h^n + h^{n+1})\left(\frac{\partial\varphi}{\partial t} - \frac{\partial^2\varphi}{\partial x^2}\right)_i^{n+1/2} +$$

$$+ a_{20}\left(\frac{\partial^2\varphi}{\partial x^2}\right)_i^{n+1/2} + a_{11}\left(\frac{\partial^2\varphi}{\partial x\partial t}\right)_i^{n+1/2} + a_{02}\left(\frac{\partial^2\varphi}{\partial t^2}\right)_i^{n+1/2} +$$

$$+ a_{30}\left(\frac{\partial^3\varphi}{\partial x^3}\right)_i^{n+1/2} + \dots + a_{03}\left(\frac{\partial^3\varphi}{\partial t^3}\right)_i^{n+1/2} +$$

+ terms of order 4, with

$$a_{20} = -\frac{1}{2}(h^{n+1} - h^n)((d_i^n)^2 + (h^n)^2 + (h^{n+1})^2) ,$$

$$a_{11} = -\frac{1}{2}(h^{n+1} - h^n)d_i^n k^n ,$$

$$a_{02} = 0 ,$$

$$a_{30} = \frac{1}{12}(h^n + h^{n+1})(5(h^n)^2 - 8h^n h^{n+1} + 5(h^{n+1})^2 - (d_i^n)^2)d_i^n - \frac{1}{2}(h^{n+1} - h^n)d_i^n k^n ,$$

$$a_{21} = -\frac{k^n}{2}\left[\frac{1}{4}(h^n + h^{n+1})(d_i^n)^2 + \frac{1}{2}(h^{n+1} - h^n)^2(h^n + h^{n+1}) + (h^{n+1} - h^n)k^n\right] ,$$

$$a_{12} = 0 ,$$

$$a_{03} = \frac{1}{24}(h^n + h^{n+1})(k^n)^3 .$$

Apply the inequalities

3.3) $$|h^{n+1} - h^n| \le \frac{1}{2} ck^n(h^n + h^{n+1}) ,$$

3.4) $$|d_i^n| \le c_0 k^n \le \frac{1}{2} c'(h^n + h^{n+1}) ,$$

which follow from (2.3), (1.5) and (2.1). Then, each of the coefficients of (3.2) satisfies

$$|a_{\alpha\beta}| \le C'k^n(h^n + h^{n+1})((h)^2 + (k)^2) ,$$

where C' is a positive constant which depends on c, c' and c_0, for $0 \le \alpha \le 3$, $0 \le \beta \le 3$, $\alpha + \beta = 2$ or 3. The coefficients of the terms of order 4 are of degree 4 in h^n, h^{n+1} and k^n and satisfy the same estimate (for which (2.3) is not even needed). Hence, after dividing (3.2) by $k^n(h^n + h^{n+1})$, we get (3.1) and the lemma is proved.

<u>Remark:</u> A Taylor expansion at the point $\tilde{P}_i^n = (h^n P_i^n + h^{n+1}P_i^{n+1})/(h^n + h^{n+1})$ is more complicated and does not give a better estimate for the truncation error.

Lemma 3.2

Under the same assumptions as in Lemma 3.1, we have

(3.5) $$|\tilde{f}_i^n - f_i^{n+1/2}| \leq C_2 k^2 \underset{m=1,2}{\text{Max}} |f|_{m,\infty,\mathcal{R}} ,$$

for all functions $f \in C^2(\bar{\mathcal{R}})$, where $\tilde{f}_i^n$ is the weighted average of f defined in (1.9), $f_i^{n+1/2} = f(P_i^{n+1/2})$ and C_2 is a constant which depends on c_0 and ℓ_0.

Proof: We have

$$\tilde{f}_i^n = (h^n f(P_i^n) + h^{n+1} f(P_i^{n+1}))/(h^n + h^{n+1}) .$$

Let $D_\xi f$ denote the derivative of f along the vector $\xi = (d_i^n, k^n)$, with $d_i^n = x_i^{n+1} - x_i^n$, and let $D_\xi^2 f$ denote the corresponding second order derivative.

A Taylor expansion at the point $P_i^{n+1/2}$ gives

$$\tilde{f}_i^n = f_i^{n+1/2} + \frac{1}{2} \frac{h^{n+1} - h^n}{h^n + h^{n+1}} D_\xi f_i^{n+1/2} + \frac{1}{8} D_\xi^2 f(P^*) ,$$

where P^* is located on the straight segment $(P_i^n P_i^{n+1})$.

But, taking account of (3.4), we have

$$|D_\xi f| = |d_i^n \frac{\partial f}{\partial x} + k^n \frac{\partial f}{\partial t}| \leq (1 + c_0) k |f|_{1,\infty,\mathcal{R}} ,$$

$$|D_\xi^2 f| \leq (1 + c_0)^2 k^2 |f|_{2,\infty,\mathcal{R}} .$$

Hence, after using (3.3), we get

$$|\tilde{f}_i^n - f_i^{n+1/2}| \leq \frac{c}{4} (1 + c_0) k^2 |f|_{1,\infty,\mathcal{R}} + \frac{1}{8} (1 + c_0)^2 k^2 |f|_{2,\infty,\mathcal{R}} ,$$

which ends the proof of the lemma, with

$$C_2 = \frac{c}{4} (1 + c_0) + \frac{1}{8} (1 + c_0)^2 \quad \text{and} \quad c = 2c_0/\ell_0 .$$

Theorem 3.1

Let the assumptions be the same as in Theorem 2.1. Assume that each straight segment $(P_i^n P_i^{n+1})$ is contained in $\mathcal{R}$ for $1 \leq i \leq I - 1$ and $t^{n+1} \leq T$. Let u and u_h be the solutions of problems $(\mathcal{P})$ and $(\mathcal{P}_h)$ respectively and assume $u \in C^4(\bar{\mathcal{R}})$. Then,

(3.6) $$\|u^n - u_h^n\|_h \le C\sqrt{t^n}\, e^{\gamma t^n/2}(h^2 + k^2) \max_{2\le m\le 4} |u|_{m,\infty,R}, \quad \text{for } 0 < t^n \le T,$$

where C is a constant which depends on c_0, ℓ_0 and λ.

Proof: Let $e_h = u - u_h$. We have $(L_h u_h)_i^n = \tilde{f}_i^n$, $(Lu)_i^{n+1/2} = f_i^{n+1/2}$. Hence

$$(L_h e_h)_i^n = (L_h u)_i^n - \tilde{f}_i^n = ((L_h u)_i^n - (Lu)_i^{n+1/2}) - (\tilde{f}_i^n - f_i^{n+1/2}) = \varepsilon_i^n .$$

Applying lemmas 3.1 and 3.2 and the inequality

$$|f|_{\mu,\infty,R} \le 2 \max_{2\le m\le 4} |u|_{m,\infty,R}, \quad \text{for } \mu = 1 \text{ and } 2 ,$$

(which follows from (1.1)), we get

$$\|\varepsilon_h^n\|_h \le \max_{1\le i\le I-1} |\varepsilon_i^n| \le C_3(h^2 + k^2) \max_{2\le m\le 4} |u|_{m,\infty,R} ,$$

with $C_3 = C_1 + 2C_2$. Then, we apply Theorem 2.1 to the function e_h and we get

$$\|e_h^n\|_h^2 \le \gamma' t^n e^{\gamma t^n} \max_{0\le\nu\le n-1} \|\varepsilon_h^\nu\|_h^2 ,$$

from which the estimate (3.6) of the theorem follows at once.

Remark 3.1

The condition that all straight segments $(P_i^n P_i^{n+1})$ must be contained in R, for $1 \le i \le I - 1$ and $t^{n+1} \le T$, is satisfied even if the boundary of R admits re-entrant corners provided $k^n \le (h^n + h^{n+1})/c_0$.

Remark 3.2: *Uniform convergence and approximation of the derivative.*

We have

$$\max_{1\le i\le I-1} |e_i^n| \le \Big(\sum_{i=1}^{I-1} |e_i^n|^2\Big)^{1/2} = \sqrt{I-1}\, \|e_h^n\|_h = O(h^{3/2}), \quad (\text{since } k \le O(h)),$$

which implies uniform convergence of u_h to u. Moreover

$$\frac{u_{i+1}^n - u_i^n}{h^n} = \frac{u(P_{i+1}^n) - u(P_i^n)}{h^n} + O(h^{1/2}) = \frac{\partial u}{\partial x}(P_i^n) + O(h^{1/2}) ,$$

which implies the uniform convergence of $\delta_x u_h$ to $\frac{\partial u}{\partial x}$.

These results are not optimal: the numerical experiments of [1] have shown quadratic uniform convergence of u_h and second order approximation of $\frac{\partial u}{\partial x}(P_i^n)$ by

$$(Du_h)_i^n = \frac{1}{h^n}\left[\frac{1}{2}(u_{i+2}^n - u_i^n) - (u_{i+2}^n - 2u_{i+1}^n + u_i^n)\right]$$

This approximation of $\frac{\partial u}{\partial x}$ was used to compute the free boundary of the Stefan problem with second order accuracy.

Remark 3.3

The foregoing results can easily be extended to the partial differential operator

$$Lu = \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)$$

where $a(x,t)$ is a given function of x and t.

4) COMMENTS

Since the publication of [1], other schemes have been proposed for the numerical solution of the heat equation on a variable mesh.

The scheme of Mori [12] [13] is very closely related to the scheme of [1]; both schemes relate the values of the approximation u_h at the six mesh-points P^n_{i-1}, P^n_i, P^n_{i+1}, P^{n+1}_{i-1}, P^{n+1}_i and P^{n+1}_{i+1}; but, the coefficients of the equations are different (compare equation (29) of [1] with equation (3.3) of [13] with $\theta = 1/2$); both schemes reduce to the Crank-Nicolson scheme in the particular case of a rectangular mesh. The scheme of Mori is derived from the application of the generalized Galerkin method using time-dependent basis functions as in Mignot [11], whereas our scheme is derived from trapezoidal finite elements in space and time. Mori proves the stability and the convergence of his scheme in the maximum norm under a severe condition of the form $k \leq C(h^n)^2$, where C is a certain constant, $C < 1$; under this condition, the scheme is of positive type and the maximum principle is valid; the proof is completed for the Stefan problem, including the computation of the free boundary, in the case of one phase [13] and in the case of two phases [14]. Let us remark that the same analysis can be applied to our scheme, since it is also of positive type under the same condition as above, just like the Crank-Nicolson scheme. However, this condition is too severe and is not satisfied in practical computations.

Another closely related scheme has been proposed and experimented by Miller, Morton and Baines [10]. This scheme is derived from the Galerkin method with time-dependent basis functions $\varphi_i(x,t)$, just like the scheme of Mori; but, at each time step, the test functions, say ψ_i, are different from the basis functions φ_i and independent of the time; more precisely, $\psi_i(x,t) = \varphi_i(x,t^{n+1})$, for $t^n < t < t^{n+1}$. An integration with respect to both variables x and t is performed in the strip $t^n < t < t^{n+1}$, as in [1]. No mathematical analysis of the method has been given.

A general class of numerical methods using finite elements of arbitrary shape in space and time has been proposed by the author [5] [6]. These methods differ from the previous ones by the property that the approximations admit a discontinuity with

respect to t at each time $t = t^n$. Unconditional stability and convergence are proved for general parabolic equations of order ≥ 2, in one or several space-dimensions. Applications of this method have been made with curved trapezoidal finite elements for the Stefan problem in one space-dimension (Bonnerot and Jamet [3]). However, since numerical quadrature formulae have been used to compute the integrals involved in the method, the mathematical results of [6] which assume exact integration *do not apply*. The numerical results show that the method is third order accurate. The same method has also been applied to multi-phase Stephan problems with appearing and disappearing phases [4]; in this case, curved triangles are used at the points which correspond to the appearance or disappearance of one of the phases.

Other methods based on straight *trapezoidal finite elements* in space and time have been studied by Lesaint and Raviart [9] and Lesaint [8]. The principle of these methods is to write the heat equation as a system of two differential equations of the first order:

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial x} = f \\[2ex] \dfrac{\partial u}{\partial x} - v = 0 \end{array}\right. \tag{4.1}$$

In [9], collocation methods are proposed for the numerical solution of (4.1); the stability and convergence of these methods are proved under the too stringent condition that the domain should be expanding, i.e. $s_1(t)$ should be a *non-increasing* function and $s_2(t)$ should be a non-decreasing function, which excludes the application to the multi-phase Stefan problem.

The methods studied in [8] are based on an integration in the whole strip $t^n < t < t^{n+1}$ as in [1], [5], [6], [10]. The functions u and v are approximated by functions u_h and v_h. It is possible to choose $v_h = \partial u_h/\partial x$; then, the corresponding methods are analogous to the methods of [1] or [5] according to whether the function u_h is continuous or discontinuous across the lines $t = t^n$. But, it is also possible to choose v_h different from $\partial u_h/\partial x$, which yields methods which are

more general than the previous ones; in particular, the approximation u_h may admit discontinuities with respect to x. Stability and convergence proofs are given, assuming exact computation of the integrals. These results do not apply to the method of [1] studied in the present paper, since this method was obtained by using the trapezoidal rule for the computation of the integrals.

Finally, let us mention that the method of [1] has been extended to the two-dimensional Stefan problem [2], to systems of non-linear hyperbolic equations with application to fluid dynamics [7] and to problems of heat diffusion with convection [15].

REFERENCES

[1] R. Bonnerot and P. Jamet, A second order finite element method for the one-dimensional Stefan problem, Internat. J. Numer. Methods Engrg., 8 (1974), pp. 811-820.

[2] ______________, Numerical computation of the free boundary for the two-dimensional Stefan problem by space-time finite elements, J. Computational Phys., 25 (1977) pp. 163-181.

[3] ______________, A third order accurate discontinuous finite element method for the one-dimensional Stefan problem, J. Computational Phys., to appear.

[4] ______________, Numerical computation of appearing and disappearing phases in the one-dimensional multiphase Stefan problem, to appear.

[5] P. Jamet, Estimations d'erreur pour l'approximation de l'équation de la chaleur dans un domaine variable par des méthodes d'éléments finis espace-temps, Journées éléments finis, Rennes, (1977).

[6] ______________, Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain, SIAM J. Numer. Anal., 15 (1978), pp. 912-928.

[7] P. Jamet and R. Bonnerot, Numerical solution of the Eulerian equations of compressible flow by a finite element method which follows the free boundary and the interfaces, J. Computational Phys., 18 (1975), pp. 21-45.

[8] P. Lesaint, Equation de la chaleur en mailles variables, Journées éléments finis, Rennes, (1978).

[9] P. Lesaint and P.-A. Raviart, Finite element collocation methods for first order systems, Math. Comp., to appear.

[10] J. V. Miller, K. W. Morton and M. J. Baines, A finite element moving boundary computation with an adaptive mesh, J. Inst. Math. Applics., 22 (1978) pp. 467-477.

[11] A. L. Mignot, Méthodes d'approximation des solutions de certains problèmes aux limites linéaires, Rend. Sem. Mat. Univ. Padova, XL (1968).

[12] M. Mori, A finite element method for solving moving boundary problems, IFIP working conference on modelling of environmental systems, Tokyo (1976), pp. 167-171.

[13] _______________, Stability and convergence of a finite element method for solving the Stefan problem, Publ. Res. Inst. Math. Sci., Kyoto Univ., 12 (1976), pp. 539-563.

[14] _______________, A finite element method for solving the two-phase Stefan problem in one space dimension, Publ. Res. Inst. Math. Sci., Kyoto Univ., 13 (1977) pp. 723-753.

[15] E. Varoglu and W. D. L. Finn, Finite elements incorporating characteristics for one-dimensional diffusion-convection equation, J. Computational Physics, to appear.

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In a previous paper devoted to the numerical solution of the Stefan problem, the author has proposed a numerical scheme to solve the heat equation on a variable mesh; this scheme is a generalization of the classical Crank-Nicolson scheme since it is identical to the Crank-Nicolson scheme in the particular case of a fixed mesh. Numerical experiements have been performed in one and two space-dimensions, but no mathematical results had been proved. In the present paper, the stability and convergence of the scheme are established together with an error estimate.

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